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# SPECTRAL THEORY OF SELF-ADJOINT ELLIPTIC OPERATORS: EIGENFUNCTION EXPANSIONS, CONVERGENCE, AND APPLICATIONS

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**ABSTRACT:** This thesis investigates spectral expansion theory for second-order elliptic differential operators in bounded domains. For a self-adjoint operator $L$ satisfying the ellipticity condition $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ ($\lambda > 0$), the spectrum is discrete, eigenfunctions form an orthonormal basis in $L^2(\Omega)$, and solutions expand as $u(x) = \sum c_i\varphi_i(x)$. Results are applied to boundary value problems, Galerkin methods, and fractional operators.

**Keywords:** elliptic operators; spectral expansions; eigenvalues; eigenfunctions; Hilbert space; self-adjoint operators; boundary value problems; Galerkin method; fractional operators.

## INTRODUCTION

Elliptic differential operators occupy a central role in mathematical physics and functional analysis. The prototype is the Laplace operator $\Delta = \sum_i \partial^2/\partial x_i^2$, generating the Laplace equation $\Delta u = 0$ and the Poisson equation $\Delta u = f(x)$, which model steady-state heat conduction, electrostatic potential, elastic deformation, and other stationary physical processes. Expressing solutions via eigenvalue–eigenfunction series has profound theoretical and computational significance. This thesis aims to establish the Spectral Expansion Theorem, analyze the discrete spectrum structure, and apply the results to numerical methods and generalized operator classes.

## THEORETICAL FOUNDATIONS

### The Second-Order Elliptic Operator.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The general second-order operator is:

$$Lu = -\sum_{i,j} a_{ij}(x)\, \partial^2 u/(\partial x_i \partial x_j) + \sum_i b_i(x)\, \partial u/\partial x_i + c(x)u \quad \ldots(1)$$

The operator is called **elliptic** if the symmetric coefficient matrix $(a_{ij})$ satisfies the uniform ellipticity condition for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$:

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad \lambda > 0 \quad \ldots(2)$$

Condition (2) ensures positive definiteness of $L$, which directly guarantees the existence and uniqueness of solutions to associated boundary value problems.

### Self-Adjointness and the Spectral Expansion Theorem.

The operator $L$ is **self-adjoint** in $H = L^2(\Omega)$ if $\langle Lu, v\rangle = \langle u, Lv\rangle$ for all $u, v \in H$. This guarantees real eigenvalues and orthogonal eigenfunctions. By the Riesz–Schauder theorem (compactness of the resolvent), the spectrum is discrete:

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty \quad \ldots(3)$$

The eigenfunctions $L\varphi_i = \lambda_i\varphi_i$ form an orthonormal basis: $\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$. ***Theorem 1 (Spectral Expansion).*** Under conditions (1)–(3), for any $f \in L^2(\Omega)$ the solution $u \in H^2(\Omega)$ is:

$$u(x) = \sum_{i=1}^{\infty} c_i\varphi_i(x), \quad c_i = \langle f, \varphi_i \rangle = \int_v f(x)\varphi_i(x)\,dx \quad \ldots(4)$$

The series converges in $L^2(\Omega)$, and in $H^2(\Omega)$ when $f \in H^2(\Omega)$, ensuring smooth solutions and control of derivatives essential for numerical computations.

**Concrete Example: Laplace Operator on $(0, \pi)$.**

For $L = -\Delta$ on $\Omega = (0,\pi)$ with Dirichlet conditions $u(0) = u(\pi) = 0$, the problem $-u'' = \lambda u$ yields:

$$\lambda_n = n^2, \quad \varphi_n(x) = \sqrt{(2/\pi)}\sin(nx), \quad n = 1, 2, 3, \ldots \quad \ldots(5)$$

Any $f \in L^2(0,\pi)$ expands as a classical Fourier sine series via (4), with $c_n = \sqrt{(2/\pi)}\int_0^{\pi} f(x)\sin(nx)\,dx$, confirming Theorem 1 in the classical setting.

**Fractional and Nonlocal Generalization.**

The framework extends to fractional elliptic operators ($0 < s < 1$):

$$L^s u(x) = \sum_i \lambda_i^s c_i\varphi_i(x), \quad 0 < s < 1 \quad \ldots(6)$$

The same eigenfunctions are retained while eigenvalues are raised to the power $s$. The operator retains positivity and self-adjointness, with applications in anomalous diffusion, financial modeling, and biological dispersal.

## APPLICATIONS AND NUMERICAL METHODS

The spectral expansion (4) underpins several numerical techniques. In the **Galerkin method**, the solution is approximated in the subspace spanned by the first $N$ eigenfunctions, yielding the error bound:

$$u^N(x) = \sum_{i=1}^{N} c_i\varphi_i(x), \quad \|u - u^N\|_{L^2} \le C\lambda_{N+1}^{-1/2}\|f\|_{L^2} \quad \ldots(7)$$

Since $\lambda_n \to +\infty$, the Galerkin approximation is spectrally convergent. The spectral basis also guides meshing and error estimation in finite element methods. Practical applications include heat conduction and thermal analysis, electrostatic and electromagnetic field computation, acoustic resonance in cavities, and elastic vibration analysis of structures.

## CHALLENGES AND FUTURE DIRECTIONS

Key challenges include: **computational complexity** — computing many eigenvalue–eigenfunction pairs in high-dimensional or geometrically complex domains is resource-intensive; **irregular coefficients** — discontinuous or degenerate $a_{ij}(x)$ require careful modification of the Riesz–Schauder argument; **nonlinear/time-dependent extensions** — coupling with nonlinear solvers or time-integration schemes introduces additional stability constraints. Future directions include integration with Physics-Informed Neural Networks (PINNs), spectral analysis on Riemannian manifolds via Weyl's law, and GPU-accelerated parallel eigensolvers for large-scale engineering problems.

## CONCLUSION

This thesis established the spectral expansion theory for second-order self-adjoint elliptic operators. The ellipticity condition (2) ensures positive definiteness and solution uniqueness; self-adjointness yields a real discrete spectrum (3) with an orthonormal eigenfunction basis; the Spectral Expansion Theorem (4) represents solutions in $L^2(\Omega)$ with spectrally convergent Galerkin approximation (7); the classical Fourier example (5) confirms the theorem; and the fractional generalization (6) extends the framework to anomalous diffusion and nonlocal problems. These results provide a robust theoretical foundation for numerical methods, practical modeling, and future research in applied mathematics and computational science.

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