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BESSEL FUNCTIONS AND BESSEL-FOURIER SERIES: THEORETICAL BASIS, HISTORICAL DEVELOPMENT, AND PRACTICAL APPLICATIONS

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Abstract: This article comprehensively analyzes the theoretical foundations, historical development, and applications of Bessel functions and Bessel-Fourier series in modern mathematics and technology. Starting from historical sources, based on the scientific works of classical and modern scientists and the results of practical research, the significance of Bessel functions, their applications in differential equations and physical processes, as well as the analytical and computational capabilities of Bessel-Fourier series are studied in depth. The article also presents methodological discussions and critical analyses.

Keywords: Bessel functions, Bessel-Fourier series, Special functions, Differential equations

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In many areas of mathematical analysis and applied mathematics, special functions, in particular Bessel functions and Bessel-Fourier series built on them, are of great importance. These functions and series are widely used, especially in modeling physical processes with cylindrical or spherical symmetry, for example, heat conduction, waves, electromagnetic fields, and many other areas. Bessel functions and Bessel-Fourier series serve as one of the main tools in the analytical and computational solution of differential equations, in the analysis of functions in cylindrical coordinates. The article extensively covers the theoretical foundations of these special functions, their historical formation, the contribution of regional and international scientists, applied research, and modern analysis methods. Also, the convergence, computational stability, and importance of Fourier series built on Bessel functions for physical models are analyzed in depth. This article aims to comprehensively cover the mathematical and practical aspects of Bessel functions and their underlying series, and includes scientific literature, modern research, and critical debates.

Literature review

Bessel functions and Bessel-Fourier series have historically been one of the most important special functions in the fields of mathematical analysis, differential equations, and applied physics, and are still at the center of current scientific research. The theoretical foundations of Bessel functions began in the 18th–19th centuries, and they initially arose from the need to analyze physical problems, in particular, processes with cylindrical and spherical symmetry. Although the initial studies were carried out by Leonard Euler, Daniel Bernoulli, and other classical scientists, the complete form of these functions and their connection with differential equations was determined by the German mathematician Friedrich Bessel in 1817. The Bessel differential equation is expressed in the form: $x^2y'' + xy' + (x^2 – \nu^2)y = 0$, and its solutions are called Bessel functions. The main types of Bessel functions – the first type ($J\nu(x)$), the second type ($Y\nu(x)$), and

the modified Bessel functions (Iν(x), Kν(x)) – occupy a central place in all classical and modern literature.

The development of classical mathematics and the achievements of the theory of differential equations played an important role in the formation of the theoretical foundations. The concept of special functions, introduced by Euler and Lagrange, and later the functions of Laplace and Legendre, along with Bessel functions, became an integral part of mathematical analysis. The works of Friedrich Bessel in the early 19th century, in particular his work "Untersuchungen über die Theorie der Bessel'schen Funktionen", are recognized as the main source for the formation of these functions. Bessel functions were initially used in astronomical problems, for example, in determining the motion of planets, as well as in studying the vibrations of cylindrical membranes. Later they also began to be widely used in the fields of heat conduction, acoustics, electromagnetic waves, and quantum mechanics. The analytical and computational properties of these functions, their asymptotic expressions, orthogonal properties, and expansion into series have become important in mathematical analysis and physics. The basic properties of Bessel functions, such as orthogonality, integral expressions, recurrence formulas, and symmetry properties, have been studied in many fundamental studies. For example, Watson's monograph "A Treatise on the Theory of Bessel Functions" is one of the most important and comprehensive sources on Bessel functions, which provides a detailed analysis of the theoretical and practical aspects of these functions.

The contribution of regional and national scientists to the development of Bessel functions and their series also occupies a special place. Research on the theory of Bessel functions and special functions in Uzbekistan, in particular, at Tashkent State University and the scientific schools under the Academy of Sciences of Uzbekistan, has been one of the leading topics. Uzbek scientists, including Professor SS Sultanov, AG Gulyomov, Sh.A. Alimov, G.T. Yakubov and others, have conducted important scientific research on the development of the theory of Bessel functions and special functions, their application to differential equations and physical models. In particular, G.T. Yakubov studied Bessel functions as orthogonal systems, and deeply analyzed their analytical and computational properties, convergence, and significance for physical models. The monographs of SS Sultanov and AG Gulyamov are considered the main sources on special functions and their role in practice, integral and series expressions of Bessel functions, their connection with differential operators, as well as their importance for modern computational methods. The studies of local scientists also widely cover the convergence of Bessel-Fourier series for smooth and non-smooth functions, properties of orthogonal systems, and analytical and computational approaches to physical problems with cylindrical symmetry.

There are many empirical studies and practical developments on the use of Bessel functions and Bessel-Fourier series in modern mathematics and practice. Bessel functions are used as a basic tool, especially in modeling physical processes expressed in cylindrical coordinates, such as heat transfer problems, waves, electromagnetic fields, acoustics, quantum mechanics, and many other areas. For example, to analyze heat transfer in a cylindrical rod or cylindrical body, the heat transfer equation is reduced to the Bessel differential equation, and its solution is expressed in terms of Bessel functions. Bessel functions and Bessel-Fourier series are also important in the processes of electromagnetic waves passing through cylindrical obstacles, in the propagation of light in optical fibers, in acoustic resonators, and in the expression of wave functions in quantum mechanics. In modern empirical research, for example, in the fields of computer mathematics and computational physics, a lot of scientific work is being carried out on computational algorithms for Bessel functions, methods for their fast and accurate calculation, discrete Bessel functions and their

series expressions. In this regard, the convergence, computational stability, discretization methods and adaptability for physical models of Bessel functions and their underlying series remain one of the main research areas.[1]

The theory of Bessel-Fourier series, that is, orthogonal series built on the basis of Bessel functions, is one of the main tools for expanding and analyzing functions in cylindrical coordinates. Any function defined in a cylindrical domain by Bessel-Fourier series can be expressed in the form of an orthogonal series in terms of Bessel functions. This approach is especially important in finding analytical solutions to differential equations with boundary and initial conditions, modeling physical processes, and developing computational algorithms. Many studies have been conducted on the convergence of Bessel-Fourier series, the behavior of the series for smooth and non-smooth functions, their integral and discrete expressions, and their suitability for physical models. For example, in the classical literature, the convergence and orthogonality of Bessel-Fourier series, their comparison with Chebyshev and Legendre series, and in modern research, the application of Bessel-Fourier series for discrete and digital signals is widely covered.

The analytical and computational properties of Bessel functions and Bessel-Fourier series, their asymptotic expressions and integral expressions have been studied in depth in many mathematical works. For example, in the monographs of classical scientists such as Watson, Olver, Abramowitz and Stegun, the analytical and computational properties of Bessel functions, their asymptotic expressions, integral and series expressions, and properties as orthogonal systems are analyzed in detail. In the study of asymptotic expressions of Bessel functions, the methods of asymptotic analysis developed, for example, by Debye and Hankel are of great importance. The connection of the integral expressions of Bessel functions with, for example, the Laplace transform, the Mellin transform, and other integral operators expands their application in physical and engineering problems. In modern research, in particular, numerical calculation algorithms for Bessel functions, their fast and accurate computational expressions, discrete Bessel functions and their series expressions are of great importance for modern computer mathematics and computational physics. In this regard, for example, Abramowitz and Stegun's Handbook of Mathematical Functions provides detailed information on practical computational formulas for Bessel functions, discrete algorithms, and their applications to physical models.[2]

One of the main controversial issues in the historical and modern study of Bessel functions and Bessel-Fourier series is their convergence properties, computational stability, and suitability for physical models. In the classical literature, the rapid convergence of Bessel-Fourier series for smooth functions, and the convergence rate and errors of the series for functions with non-smooth or limited smoothness, have been widely discussed. In modern research, especially the application of Bessel-Fourier series for digital signals and discrete data, their discretization and numerical computing algorithms, Bessel functions and the stability of the series based on them as discrete orthogonal systems, and their suitability for physical models remain one of the main controversial areas. For example, there are important scientific debates on the application of Bessel functions as discrete orthogonal systems, their comparison with Chebyshev, Legendre, and other classical orthogonal systems, the advantages and disadvantages of Bessel-Fourier series in modern computational mathematics and digital signal analysis, and their optimality for physical models.[3]

There are also many empirical studies and practical developments on the application of Bessel functions and Bessel-Fourier series in modern physics and engineering. For example, in the propagation of electromagnetic waves through cylindrical obstacles, light propagation in optical fibers, vibration processes in acoustic resonators, the expression of wave functions in quantum

mechanics, heat conduction, and other engineering problems, Bessel functions and Bessel-Fourier series are used as the main tools. In the fields of modern computer technology and computational physics, many scientific works are being carried out on numerical calculation algorithms for Bessel functions, their fast and accurate computational expressions, and the optimality of discrete Bessel functions and series based on them for physical models. In this regard, for example, in the work "Handbook of Mathematical Functions" by Abramowitz and Stegun, detailed information on practical calculation formulas for Bessel functions, discrete algorithms, and their application to physical models is provided.[4]

Many modern studies are being conducted on the application of Bessel functions and Bessel-Fourier series in modern mathematics and practice, their analytical and computational capabilities, their suitability for physical models, and numerical calculation algorithms. For example, in the fields of modern computer mathematics and computational physics, there are important scientific works on numerical calculation algorithms of Bessel functions, discretization methods, suitability for physical models, convergence and computational stability of discrete Bessel functions and series based on them. In this regard, for example, there are important scientific discussions on the use of Bessel functions as discrete orthogonal systems, their comparison with Chebyshev, Legendre, and other classical orthogonal systems, the advantages and disadvantages of Bessel-Fourier series in modern computational mathematics and digital signal analysis, and their optimality for physical models.[5]

In the studies conducted on Bessel functions and Bessel-Fourier series, as well as the development of their generalized and modified forms, such as spherical Bessel functions, modified Bessel functions, Chebyshev-Bessel functions, Legendre-Bessel functions, and other special functions, one of the important research areas remains. The suitability of generalized Bessel functions and series based on them for physical and engineering models, their analytical and computational properties, are widely covered in modern research. For example, modified Bessel functions are used in modeling heat transfer, diffusion, and other physical processes, while spherical Bessel functions are used as a main tool in analyzing processes with spherical symmetry. In modern research, many scientific works are also being carried out on the integral and series expressions of Bessel functions, their modern computational algorithms, their suitability and optimality for physical models, discretization methods, and their application to digital signals.[6]

In the process of critical analysis, the issues of convergence, computational stability, and optimality for physical models of Bessel functions and Bessel-Fourier series for smooth and non-smooth functions are one of the main controversial areas. In the classical literature, the rapid convergence of Bessel-Fourier series for smooth functions, and the problems of convergence speed and errors of the series for non-smooth or functions with limited smoothness are widely discussed. In modern research, especially the application of Bessel-Fourier series for digital signals and discrete data, their discretization and digital computing algorithms, Bessel functions as discrete orthogonal systems and the stability of the series based on them, and their suitability for physical models remain one of the main controversial areas. There are, for example, important scientific debates on the use of Bessel functions as discrete orthogonal systems, their comparison with Chebyshev, Legendre and other classical orthogonal systems, the advantages and disadvantages of Bessel-Fourier series in modern computational mathematics and digital signal analysis, and their optimality for physical models. In general, Bessel functions and Bessel-Fourier series are the main tools in mathematical analysis, differential equations, physical and engineering models, and their theoretical, analytical and practical properties are still the focus of scientific research. Modern

research, empirical work and critical analysis further increase the mathematical and practical significance of these special functions.

Conclusion

According to the results of the above analysis, Bessel functions and Bessel-Fourier series retain their importance as the most important special functions in mathematical analysis, differential equations, physics and engineering. Fundamental and modern research on their theoretical foundations, historical development and practical application has further increased the scientific and practical significance of these functions. Bessel functions play a key role in expressing solutions to differential equations, modeling physical processes, developing computational algorithms and solving modern technological problems. Bessel-Fourier series are an important tool for extending functions in cylindrical spheres, analyzing physical models and ensuring computational stability. As a result of modern research, empirical work and critical analysis, the issues of convergence, computational stability and optimality of Bessel functions and series based on them for physical models are being studied in more depth. At the same time, new scientific directions are being formed in the field of generalized and modified types of Bessel functions and Bessel-Fourier series, their application to modern computational algorithms and digital signals. In general, Bessel functions and Bessel-Fourier series will continue to be the main source of relevant scientific research and practical developments in various fields of mathematics and applied sciences in the future.

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