BG Pulse Journal USA
ISSN:1947-2536 http://www.bgpulseusa.com/index.php/bg
BG PULSE JOURNAL

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE CAUCHY PROBLEM WITH INTEGRABLE COEFFICIENTS

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Doi: https://doi.org/10.5281/zenodo.21350472

Abstract: This thesis deals with the existence and uniqueness of solutions to the Cauchy problem for ordinary differential equations (ODEs). First, the local existence of solutions is stated under the Carathéodory conditions (measurability in time and continuity in the state variable). Then, sufficient conditions for global existence and uniqueness are given, namely boundedness and Lipschitz continuity of the function f(t,y).

Key words: Cauchy problem, Lipschitz, Picard mapping, Banach fixed point theorem, contraction mapping, norm, measurable function, local solution, global solution.

In order to prove the existence and uniqueness of solutions to the Cauchy problem, we will use the Banach fixed point theorem (also known as the contraction mapping theorem). This theorem provides a powerful tool for solving differential equations by converting them into fixed point problems.

Theorem (Banach fixed point theorem).

Let (X,d) *be a complete metric space and let* $F:X\to X$ *be a contraction. That is, there exists a constant* $k\in[0,1)$ *such that*

$$d(F(x),F(y))\le k\cdot d(x,y) \qquad \text{for all } x,y\in X.$$

Then:

1. *F has exactly one fixed point* $x^*\in X$ *(i.e.,* $F(x^*)=x^*$*).*
2. *For any initial point* $x_0\in X$*, the sequence defined by*

$$x_{n+1}=F(x_n), \quad n=0,1,2,\dots$$

converges to x^* .

The Banach fixed point theorem **[3]** is applied to the Picard map to prove existence and uniqueness of the solution.

We now consider the fundamental Cauchy problem for ordinary differential equations. We assume that the function $f:\mathbb{R}^{n+1}\to\mathbb{R}^n$ satisfies the following Carathéodory conditions **[1]**:

1) For every $y\in\mathbb{R}^n$, the function $f(t,y)$ is measurable with respect to t.
For every $t\in\mathbb{R}$, the function $f(t,y)$ is continuous with respect to y.
2) For every compact $K\subset\mathbb{R}^{n+1}$ there exist constants M_K, L_K such that
$|f(t,y)|\le M_K$, $\quad |f(t,x)-f(t,y)|\le L_K|x-y|$ for all $(t,x),(t,y)\in K$.

Theorem (Cauchy).

Let $f(t,y)$ be a function and let $y(t_0)=y_0$ be the starting condition.

- *Local solution. If f satisfies 1) and 2) then there exists a small $\varepsilon>0$ such that a solution exists on $[t_0,t_0+\varepsilon]$.*

- *Global solution* ***[2].*** *If, in addition, f is bounded and Lipschitz in y for all t,x,y, i.e.,*

$$|f(t,y)|\leq C \quad and \quad |f(t,x)-f(t,y)|\leq L|x-y|,$$

then:

For any $T>t_0$, a unique solution exists on the whole interval $[t_0,T]$.

Note: This solution depends continuously on the starting value y_0.

We show that global existence and uniqueness of solutions still hold if condition 2) is replaced by the following weaker one:

- *There exist integrable functions $\varphi, \psi:[t_0, T]\to\mathbb{R}_+$ satisfying*

$$|f(t,y)|\leq\varphi(t), \quad |f(t,x)-f(t,y)|\leq\psi(t)|x-y|$$

for all $(t,x),(t,y)\in\mathbb{R}\times\mathbb{R}^n$.

Proof of the main result: Let X be the set of all continuous functions from $[t_0,T]$ to $\mathbb{R}^n$. Following **[4],** define a norm on X by

$$/\!/ w /\!/ _* = \max_{t\in[t_0,T]} \exp\left(-2\int_{t_0}^{t}\psi(s)ds\right)|w(t)|.$$

This norm is equivalent to the usual maximum norm.

Define the Picard map F by

$$F(w)(t)=y_0+\int_{t_0}^{t} f(s,w(s))ds, \qquad t\in[t_0,T].$$

The map is well defined because $f(s,w(s))$ is measurable and $|f(s,w(s))|\leq\varphi(s)$ with φ integrable. Take any two functions $w,w'\in X$ and let $\delta=/\!/ w-w' /\!/ _*$. From the definition of the norm we get

$$|w(s)-w'(s)|\leq\delta\exp\left(2\int_{t_0}^{s}\psi(r)dr\right).$$

Now compute:

$$\&\exp\left(-2\int_{t_0}^{t}\psi(s)ds\right)|F(w)(t)-F(w')(t)|$$
$$\&\leq\exp\left(-2\int_{t_0}^{t}\psi(s)ds\right)\int_{t_0}^{t}\psi(s)|w(s)-w'(s)|ds$$
$$\&\leq\exp\left(-2\int_{t_0}^{t}\psi(s)ds\right)\int_{t_0}^{t}\psi(s)\cdot\ \delta\exp\left(2\int_{t_0}^{s}\psi(r)dr\right)ds.$$

Since

$$\frac{d}{ds}\exp\left(2\int_{t_0}^{s}\psi(r)dr\right)=2\psi(s)\exp\left(2\int_{t_0}^{s}\psi(r)dr\right),$$

we have

$$\int_{t_0}^{t}\psi(s)\exp\left(2\int_{t_0}^{s}\psi(r)dr\right)ds=\frac{1}{2}\left(\exp\left(2\int_{t_0}^{t}\psi(r)dr\right)-1\right).$$

Therefore,

$$\exp\left(-2\int_{t_0}^{t}\psi(s)ds\right)|F(w)(t)-F(w')(t)|\leq\frac{\delta}{2}\left(1-\exp\left(-2\int_{t_0}^{t}\psi(r)dr\right)\right)\leq\frac{\delta}{2}.$$

Taking the maximum over t gives

$$\|F(w)-F(w')\|_{*}\leq\frac{1}{2}\|w-w'\|_{*}.$$

So F is a contraction (see **[3,4]**).

By the Banach fixed point theorem, there exists a unique $y\in X$ such that $F(y)=y$, i.e.

$$y(t)=y_0+\int_{t_0}^{t}f(s,y(s))ds.$$

This y is the unique global solution of $\dot{y}=f(t,y)$, $y(t_0)=y_0$ on $[t_0,T]$. The solution depends continuously on y_0 because F depends linearly on y_0 and the contraction constant is less than 1. ∎

These results are essential for differential equations with measurable time dependence, such as those arising in control theory.

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