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THE PROBLEM OF BOUNDED SOLUTIONS FOR N-TH ORDER VECTOR-MATRIX DIFFERENTIAL EQUATIONS

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Abstract. This article investigates the problems of existence and uniqueness of bounded solutions for a system of n-th order nonlinear vector-matrix differential equations. In the course of the study, sufficient conditions ensuring the existence, uniqueness, and boundedness of solutions of the considered system are established and formulated in the form of main theorems. The obtained results are justified on the basis of Banach's fixed point principle and the method of integral equations. Furthermore, the significance of the derived estimates in the investigation of qualitative properties of solutions to differential systems is demonstrated.

Keywords: differential equation, vector–matrix system, bounded solution, Banach theorem, integral equation, Green's function, Lipschitz condition, fixed-point principle.

Introduction. Mathematical models of multidimensional dynamical systems are often represented by vector-matrix differential equations. One of the main problems in the study of such equations is the determination of the existence, uniqueness, and boundedness of their solutions, since these issues play an important role in analyzing the stability of systems and their practical applicability.

Literature Review. In this direction, the problem of bounded solutions for higher-order nonlinear vector-matrix differential equations was extensively investigated by Ivan D. Kostrub and Alexander I. Perov [1], [3]. In these studies, the equations were mainly considered in finite-dimensional Banach and Hilbert spaces. Although the existing scientific results demonstrate the significant development of this theory, the problem of determining the existence and properties of bounded solutions under more general conditions remains relevant. The present work is devoted to the investigation of these issues.

Research Methodology. It is known that n-th order nonlinear differential equations can be reduced to a first-order vector-matrix system and investigated in the following form:

$$X^{(n)}(t)+\sum_{k=0}^{n-1} A_k(t)X^k(t)=F\left(t,X(t),X(t),X(t),...,X^{(n-1)}(t)\right),\ t \quad t_0 \qquad (1)$$

Here, $X(y)$ ^{m}, $A_k(t)-$ are continuous matrices of order m m tartibli and F is a nonlinear vector-valued function.

For simplification, equation (1) is transformed into a first-order system by introducing the substitutions

$$Y_1 = X,\ Y_2 = X,\ Y_3 = X,\ ...,\ Y_n = X^{(n-1)}.$$

As a result, the following $N = n$ $m -$ dimensional vector-matrix differential system is obtained:

$$Y(t) = A(t)Y(t) + F(t,Y(t)),\ t \quad t_0. \qquad (2)$$

The main objective of this work is to establish sufficient conditions ensuring that system (2) possesses bounded solutions on the interval $[t_0,+\ )$ and to provide rigorous proofs of these conditions. Before presenting the proofs, we introduce the functional space and the main assumptions used throughout the paper.

Definition 1. Let B be the space of all continuous and bounded vector-valued functions $Y(t)$ defined on the interval $[t_0,+\ )$. The norm in this space is defined by

$$\|Y(t)\|_B = \sup_{t\ t_0}\|Y(t)\|.$$

Assumption 1. For the corresponding homogeneous linear system $Y'(t) = A(t)Y(t)$ suppose that its Green function satisfies the estimate

$$\int_{t_0}^{+}\|G(t,s)\|ds \quad K, \quad \forall t \quad t_0,$$

where $K > 0$ is a constant.

Assumption 2. Suppose that the function $F(t,\mathrm{Y})$ satisfies the Lipschitz condition with respect to the variable Y, namely,

$$\|F(t,\mathrm{Y}_1)-F(t,Y_2)\| \quad L\|Y_1-Y_2\|, \quad \|F(t,0)\| \quad M,$$

where $L > 0$ and $M > 0$ are constants.

Theorem 1. If Assumptions 1 and 2 hold and the inequality $K \quad L < 1$ is satisfied, then system (2) possesses a unique bounded solution in the space B.

Proof. The proof of this theorem is based on Banach's fixed point theorem. First, system (2) is transformed into the following integral equation. Using the properties of the Green function, every solution of system (2) satisfies

$$Y(t) = \int_{t_0}^{+} G(t,s)F(s,Y(s))ds. \qquad (3)$$

Next, define the following operator in the space B:

$$(TY)(t) = \int_{t_0}^{+} G(t,s)F(s,Y(s))ds.$$

We show that the operator T maps the space B into itself and is a contraction operator.

First, it is necessary to prove that $T : B \to B$.

Take an arbitrary function $Y \quad B$. By the Lipschitz condition, we obtain

$$\|F(s,Y(s))\| \quad \|F(s,Y(s))-F(s,0)\|+\|F(s,0)\| \quad L\|Y(s)\|+M.$$

Since $Y \quad B$, it follows that $\|Y(s)\| \quad \|Y\|_B$.

Hence,

$$\|F(s,Y(s))\| \quad \|F(s,Y(s))-F(s,0)\|+\|F(s,0)\| \quad L\|Y(s)\|+M.$$

Now let us estimate the norm of $(TY)(t)$:

$$\|(TY)(t)\| \int_{t_0}^{+} \|G(t,s)\|\|F(s,Y(s))\|ds \quad (L\|Y\|_B + M)\int_{t_0}^{+} \|G(t,s)\|ds.$$

By Assumption 1, $\int_{t_0}^{+} \|G(t,s)\|ds \quad K.$

Therefore, $\|(TY)(t)\| \quad K(L\|Y\|_B + M)$. This estimate is valid for all $t \quad t_0$. Hence,

$\|(TY)(t)\| \quad K(L\|Y\|_B + M) < \quad .$

Thus, $TY \quad B$ that is, the operator maps the space B into itself.

In the second step, we prove that the operator T is contractive. For arbitrary $\forall Y_1, Y_2 \quad B$, we have

$$\|(TY_1)(t) - (TY_2)(t)\| = \left\| \int_{t_0}^{+} G(t,s)\left(F(s,Y(s)) - F(s,Y_2(s))\right)ds \right\|.$$

By the Lipschitz condition, $\|F(s,Y_1) - F(s,Y_2)\| \quad L\|Y_1 - Y_2\|$. Therefore,

$$\|(TY_1)(t) - (TY_2)(t)\| \quad L\|Y_1 - Y_2\|_B \int_{t_0}^{+} \|G(t,s)\|ds \quad KL\|Y_1 - Y_2\|_B.$$

Therefore, $\|TY_1 - TY_2\|_B \quad q\|Y_1 - Y_2\|B, \quad q = KL < 1.$ This shows that the operator T is a contraction operator.

According to Banach's fixed point theorem, the contraction operator T in the complete metric space B possesses a unique fixed point. That is, there exists a unique function $Y \quad B$ satisfying the equation $TY = Y$. This function is the unique solution of the integral equation (3), and consequently, it is also the unique bounded solution of differential system (2) [2].

Lemma 1. (Sufficient condition for boundedness)

Suppose that Assumptions 1 and 2 hold and the inequality $KL < 1$ is satisfied. Then the unique bounded solution $Y(t)$ of system (2) satisfies the estimate $\|Y\|_B \quad \frac{KM}{1-KL}$.

Proof. As shown in the proof of Theorem 1, every bounded solution of differential system (2) satisfies the following integral equation:

$$Y(t) = \int_{t_0}^{+} G(t,s)F(s,Y(s))ds.$$

Using this equality, let us estimate the norm of the solution. By the properties of the norm,

$$\|Y(t)\| \int_{t_0}^{+} \|G(t,s)\| \|F(s,Y(s))\|ds. \qquad (4)$$

According to Assumption 2, the function $F(t,Y)$ satisfies the Lipschitz condition, and therefore

$$\|F(s,T(s))\| \quad \|F(s,Y(s)) - F(s,0)\| \quad L\|Y(s)\| + M \ . \qquad (5)$$

Since $Y \quad B$, we have $\|Y(s)\| \quad \|Y\|_B$.

Hence,

$$\|F(s,Y(s))\| \quad L\|Y(s)\| + M. \qquad (6)$$

Substituting this estimate into the previous inequality, we obtain

$$\|Y(t)\| \quad L\|Y(s)\| + M \int_{t_0}^{+} \|G(t,s)\| ds. \qquad (7)$$

By Assumption 1,

$$\int_{t_0}^{+} \|G(t,s)\| ds \quad K \qquad \|Y(t)\| \quad K\left(L\|Y\|_B + M\right).$$

Taking the supremum, we obtain the inequalities

$$\|Y\|_B \quad K\left(L\|Y\|_B + M\right) \qquad (1 - K\,L)\|Y\|_B \quad KM.$$

Since $K\,L < 1$, it follows that $\|Y\|_B \quad \dfrac{K\,M}{1 - K\,L}$ Thus, the lemma is proved.

Corollary 1. Suppose that Assumptions 1 and 2 hold and, in addition, $F(t,0) = 0$. Then the unique bounded solution of system (2) is the trivial solution $Y(t) \quad 0$.

Proof. In this case, $M \quad 0$. By Lemma 1, $\|Y\|_B \quad 0$. Hence, $\|Y\| \quad 0$. The corollary is proved.

Example 1. Consider the differential equation

$$X\ (t) + 2X\ (t) + 5X(t) = \frac{1}{3}\sin X(t),\ t \quad 0.$$

We verify the assumptions of Theorem 1 and show that the system possesses a unique bounded solution in the space B.

To reduce the given equation to a first-order system, introduce the substitutions $Y_1 = X$, $Y_2 = X$.

Then the equation takes the form

$$Y_1 = Y_2$$

$$Y_2 = -2X - 5X + \frac{1}{3}\sin X = -2Y_2 - 5Y_1 + \frac{1}{3}\sin Y_1 .$$

This system can be written in matrix form as

$$Y = \begin{matrix} 0 & 1 \\ -5 & -2 \end{matrix} Y + \begin{matrix} 0 \\ \frac{1}{3}\sin Y_1 \end{matrix}.$$

Now let us determine the eigenvalues of the matrix A:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -5 & -2 - \lambda \end{vmatrix} = \lambda^2 + 2\lambda + 5 = 0$$

Hence, $\lambda_{1,2} = -1 \pm 2i$.

Therefore, for the Green function of the system there exists an estimate $\int_0^+ \|G(t,s)\|ds \quad K, (K \approx 1)$.

Since the nonlinear term satisfies the Lipschitz condition with constant $L = \frac{1}{3}$ we obtain $K L \approx 1 \ \frac{1}{3} = \frac{1}{3} < 1$.

Thus, the assumptions of Theorem 1 are satisfied, and the system possesses a unique bounded solution in the space B.

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