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FUNDAMENTAL THEORY OF METRIC SPACES AND THEIR APPLICATIONS IN FUNCTIONAL ANALYSIS

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Abstract: This paper systematically investigates the concepts of metric spaces and metrics (distance functions), which are fundamental pillars of modern mathematical analysis and functional analysis. The axioms of a metric, the topological structure of metric spaces, the convergence of sequences, and the theory of complete metric spaces are thoroughly analyzed. Specifically, metrics defined on finite-dimensional spaces $\mathbb{R}^n$, sequence spaces l_p, the space of continuous functions C[a,b], and Lebesgue integrable function spaces $L_p(X,\mu)$ are examined alongside the role of Hölder and Minkowski inequalities. Finally, the contraction mapping principle (Banach fixed-point theorem) in complete metric spaces and its application to the theory of integral equations are presented.

Keywords: metric space, metric axioms, triangle inequality, Cauchy sequence, complete metric space, function space, Banach theorem, fixed point

In the classical course of mathematical analysis, concepts such as limit, continuity, convergence, and compactness are initially introduced on the set of real numbers $\mathbb{R}$ or in the n-dimensional Euclidean space $\mathbb{R}^n$ based on the notion of distance between two points. At the end of the 19th century and the beginning of the 20th century, as a result of the rapid development of the theory of differential and integral equations as well as the calculus of variations, there arose a necessity to define the notions of proximity and limiting processes in abstract sets whose elements consist not only of numbers or points, but also of functions, sequences, sets, or operators.

In 1906, the French mathematician Maurice Fréchet, in his famous doctoral dissertation, axiomatically introduced for the first time the concept of “distance” (later termed a metric by Felix Hausdorff) on a set consisting of elements of arbitrary nature. This idea made it possible to apply the methods of mathematical analysis and geometry to function spaces, laying the foundation for the emergence of functional analysis and general topology as independent disciplines.

The aim of this paper is to analyze the axiomatic foundations of metric spaces, the methods of introducing metrics on sets of various mathematical objects, the equivalence of metrics, and the concept of completeness of a space, as well as to reveal their significance in modern operator theory.

Metric Axioms and Basic Concepts

Let X be an arbitrary non-empty set ($X\neq\varnothing$).

Definition 1. If a function $\rho:X\times X\to[0,+\infty)$ assigns to each pair of elements $(x,y)\in X\times X$ of the set X a unique non-negative real number $\rho(x,y)\in\mathbb{R}$ satisfying the following axioms, then the function ρ is called a metric (or distance function) on X, and the pair (X,ρ) is called a metric space:

1. **Identity (indiscernibility) axiom:** $\rho(x,y)=0\Leftrightarrow x=y$ $(\forall x,y\in X)$;
2. **Symmetry axiom:** $\rho(x,y)=\rho(y,x)$ $(\forall x,y\in X)$;
3. **Triangle inequality:** $\rho(x,z)\le\rho(x,y)+\rho(y,z)$ $(\forall x,y,z\in X)$.

The elements $x\in X$ of a metric space are usually referred to as its points, and the number $\rho(x,y)$ is called the distance between the points x and y.

Proposition 1. In any metric space (X,ρ), the reverse triangle inequality holds for all points $x,y,z\in X$: $|\rho(x,z)-\rho(y,z)|\le\rho(x,y)$.

Proof. By the triangle axiom, $\rho(x,z)\le\rho(x,y)+\rho(y,z)$, which implies $\rho(x,z)-\rho(y,z)\le\rho(x,y)$. Similarly, interchanging x and y and using the symmetry axiom, we obtain $\rho(y,z)-\rho(x,z)\le\rho(y,x)=\rho(x,y)$. Combining these two inequalities yields relation (2.1). □

Example 1. Let (X,d) be an arbitrary metric space (for instance, $X=\mathbb{R}$ with $d(x,y)=|x-y|$). Prove that the function $\rho(x,y)$ defined on X by the formula

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)},\quad \forall x,y\in X,$$

satisfies all three metric axioms and defines a metric on X.

Solution. We consecutively verify that the given function $\rho(x,y)$ satisfies all three axioms of Definition 1:

1) Identity axiom: Since $d(x,y)\ge0$, it follows that $\rho(x,y)\ge0$. Furthermore,

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)}=0\Leftrightarrow d(x,y)=0\Leftrightarrow x=y,$$

because d is a metric on X.

2) Symmetry axiom: Since $d(x,y)=d(y,x)$, we have

$$\rho(x,y)=\frac{d(x,y)}{1+d(x,y)}=\frac{d(y,x)}{1+d(y,x)}=\rho(y,x).$$

3) Triangle inequality: To prove $\rho(x,z)\le\rho(x,y)+\rho(y,z)$ for arbitrary $x,y,z\in X$, consider the auxiliary function $f(t)=\frac{t}{1+t}=1-\frac{1}{1+t}$ on the interval $[0,+\infty)$. Since its derivative is

$$f'(t)=\frac{1}{(1+t)^2}>0,\quad \forall t\in[0,+\infty),$$

the function f(t) is strictly increasing on $[0,+\infty)$. Using the triangle inequality $d(x,z)\le d(x,y)+d(y,z)$ for the metric d and the monotonicity of f(t), we obtain:

$$\begin{aligned}\rho(x,z)&=f(d(x,z))\le f(d(x,y)+d(y,z))=\frac{d(x,y)+d(y,z)}{1+d(x,y)+d(y,z)}\\&=\frac{d(x,y)}{1+d(x,y)+d(y,z)}+\frac{d(y,z)}{1+d(x,y)+d(y,z)}\\&\le\frac{d(x,y)}{1+d(x,y)}+\frac{d(y,z)}{1+d(y,z)}=\rho(x,y)+\rho(y,z).\end{aligned}$$

Thus, all three metric axioms are satisfied, and $\rho(x,y)$ defines a bounded metric ($\rho(x,y)<1$) on X.

Remark 1. If in Axiom 1 of Definition 1 only the condition $x=y \Rightarrow \rho(x,y)=0$ holds, while $\rho(x,y)=0$ may also occur for some elements $x \neq y$, then such a function ρ is called a **pseudometric** (or semimetric), and (X,ρ) is called a pseudometric space. Pseudometric spaces play an important role in measure and Lebesgue integration theory.

Classification of Classical and Functional Metric Spaces

By introducing different metrics on the same set X, one can construct metric spaces whose topological and geometric properties differ fundamentally from one another. Below we consider the most widely used metric spaces in mathematical and functional analysis.

Discrete Metric Space

On an arbitrary non-empty set $X \neq \varnothing$, we introduce a distance according to the following rule:

$$\rho(x,y)=\begin{cases}0, & \text{if } x=y,\\ 1, & \text{if } x\neq y.\end{cases}$$

This function ρ satisfies all the metric axioms and is called the discrete metric.

Finite-Dimensional Coordinate Space ($\mathbb{R}^n$)

On the set $\mathbb{R}^n$ consisting of ordered n-tuples of real numbers $x=(x_1,x_2,\ldots,x_n)$ and $y=(y_1,y_2,\ldots,y_n)$, the following basic metrics are defined:

1. Euclidean metric **(ρ_2):**

$$\rho_2(x,y)=\left(\sum_{k=1}^{n}|x_k-y_k|^2\right)^{1/2}.$$

Here, the triangle inequality follows from the Cauchy–Bunyakovsky–Schwarz inequality.

2. Manhattan (lattice or octahedral) metric (ρ_1):

$$\rho_1(x,y)=\sum_{k=1}^{n}|x_k-y_k|.$$

This metric is widely used as a lattice norm in the theory of Schrödinger operators and convolution operators defined on discrete lattices ($\mathbb{Z}^n$).

3. Chebyshev (cubic or maximum) metric (ρ_∞):

$$\rho_\infty(x,y)=\max_{1\le k\le n}|x_k-y_k|.$$

4. Minkowski metric (ρ_p, $1\le p<\infty$):

$$\rho_p(x,y)=\left(\sum_{k=1}^{n}|x_k-y_k|^p\right)^{1/p}.$$

Sequence Spaces (l_p and l_∞)

Metric spaces defined on sets of infinite sequences $x=(x_1,x_2,\ldots,x_k,\ldots)$ include:

- The space l_p **($1\le p<\infty$):** The set of all sequences satisfying $\sum_{k=1}^{\infty}|x_k|^p<\infty$, where the metric is defined by:

$$\rho_p(x,y)=\left(\sum_{k=1}^{\infty}|x_k-y_k|^p\right)^{1/p}.$$

The validity of the triangle inequality is based on Minkowski's inequality for infinite series.

- The space of bounded sequences l_∞: The set of sequences satisfying $\sup_{k\in\mathbb{N}}|x_k|<\infty$:

$$\rho_\infty(x,y)=\sup_{k\in\mathbb{N}}|x_k-y_k|.$$

Spaces of Continuous and Integrable Functions

1. The space $C[a,b]$: On the set of all real-valued (or complex-valued) functions $x(t),y(t)$ defined and continuous on the interval $[a,b]$, the uniform (Chebyshev) metric is introduced as follows:

$$\rho_C(x,y)=\max_{t\in[a,b]}|x(t)-y(t)|.$$

In this space, convergence of a sequence with respect to the metric coincides with the uniform convergence of the sequence of functions on the interval $[a,b]$.

2. Lebesgue spaces $L_p(X,\mu)$ $(1\le p<\infty)$: Let (X,A,μ) be a measure space. On the set of equivalence classes of functions satisfying $\int_X |f(t)|^p\,d\mu(t)<\infty$ and equal μ-almost everywhere, the metric is defined by the formula:

$$\rho_{L_p}(f,g)=\left(\int_X |f(t)-g(t)|^p\,d\mu(t)\right)^{1/p}.$$

In particular, when $p=2$, one obtains the Hilbert space $L_2(X,\mu)$, which serves as the fundamental space in quantum mechanics and the spectral theory of operators.

Topology, Convergence, and Completeness in Metric Spaces

The concept of a metric naturally induces a topological structure on a set—namely, a family of open and closed sets. The open ball centered at a point $x_0\in X$ with radius $r>0$ is defined as $B(x_0,r)=\{x\in X:\rho(x_0,x)<r\}$, and the closed ball is defined as $\overline{B}(x_0,r)=\{x\in X:\rho(x_0,x)\le r\}$.

Definition 2. A sequence of points $\{x_n\}_{n=1}^{\infty}\subset X$ in a metric space is said to converge to a point $x\in X$ if $\lim_{n\to\infty}\rho(x_n,x)=0$, and is denoted by $x_n\to x$ $(n\to\infty)$.

Definition 3. If for every $\varepsilon>0$ there exists a natural number $N(\varepsilon)\in\mathbb{N}$ such that for all indices $n,m\ge N(\varepsilon)$ the inequality $\rho(x_n,x_m)<\varepsilon$ holds, then the sequence $\{x_n\}_{n=1}^{\infty}$ is called a fundamental sequence (or Cauchy sequence).

Every convergent sequence is fundamental: indeed, if $x_n\to x$, then by the triangle inequality $\rho(x_n,x_m)\le\rho(x_n,x)+\rho(x,x_m)\to 0$ as $n,m\to\infty$. However, the converse does not always hold in an arbitrary metric space.

Definition 4. If every fundamental (Cauchy) sequence in a metric space (X,ρ) converges to some point $x\in X$ belonging to the space itself, then (X,ρ) is called a complete metric space.

Example 2. In the metric space $C[0,1]$, consider the sequence of functions given by

$$y_n(t)=t^n-t^{2n},\quad t\in[0,1],\quad n\in\mathbb{N}.$$

Does the sequence $\{y_n(t)\}_{n=1}^{\infty}$ converge in the metric space $C[0,1]$?

Solution. First, we determine the pointwise limit of the sequence $\{y_n(t)\}$ on the interval $[0,1]$:

- If $t\in[0,1)$, then $\lim_{n\to\infty}t^n=0$ and $\lim_{n\to\infty}t^{2n}=0$, which yields $\lim_{n\to\infty}y_n(t)=0$.

- If $t=1$, then $y_n(1)=1^n-1^{2n}=1-1=0$, so $\lim_{n\to\infty}y_n(1)=0$.

Hence, the pointwise limit of $\{y_n(t)\}$ on $[0,1]$ is the zero function:

$y_0(t)\equiv 0\in C[0,1]$.

If $\{y_n\}$ converges in the metric space $C[0,1]$, its limit must coincide with the pointwise limit $y_0(t)\equiv 0$. This holds if and only if $\lim_{n\to\infty}\rho_C(y_n,0)=0$.

By the definition of the uniform metric on $C[0,1]$, we have:

$$\rho_C(y_n,0)=\max_{t\in[0,1]}|y_n(t)-0|=\max_{t\in[0,1]}(t^n-t^{2n}),$$

since $t^n\geq t^{2n}$ (i.e., $y_n(t)\geq 0$) for all $t\in[0,1]$.

To find the maximum value of $y_n(t)$ on $[0,1]$, we differentiate $y_n(t)$ with respect to t and set the derivative to zero:

$$y_n'(t)=nt^{n-1}-2nt^{2n-1}=nt^{n-1}(1-2t^n)=0.$$

Solving for the stationary point in the open interval $(0,1)$ gives:

$$1-2t^n=0\Rightarrow t^n=\frac{1}{2}\Rightarrow t_{max}=\left(\frac{1}{2}\right)^{1/n}=2^{-1/n}\in(0,1).$$

Evaluating $y_n(t)$ at the maximum point $t_{max}=2^{-1/n}$, we obtain:

$$\rho_C(y_n,0)=y_n\left(2^{-1/n}\right)=\left(2^{-1/n}\right)^n-\left(2^{-1/n}\right)^{2n}=\frac{1}{2}-\left(\frac{1}{2}\right)^2=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}.$$

Taking the limit as $n\to\infty$:

$$\lim_{n\to\infty}\rho_C(y_n,0)=\lim_{n\to\infty}\frac{1}{4}=\frac{1}{4}\neq 0.$$

Although $\{y_n(t)\}$ converges pointwise to $y_0(t)\equiv 0$ on $[0,1]$, it does not converge uniformly since $\rho_C(y_n,0)=\frac{1}{4}\neq 0$.Answer: The sequence $y_n(t)=t^n-t^{2n}$ is not convergent in the metric space $C[0,1]$.

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