BG Pulse Journal USA
ISSN:1947-2536 http://www.bgpulseusa.com/index.php/bg
BG PULSE JOURNAL

STATEMENT OF NON- LOCAL BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

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Annotation. This paper considers the formulation of nonlocal boundary value problems for third-order hyperbolic equations. The characteristics of nonlocal conditions are explored, key definitions are presented, and a model problem is formulated. The well-posedness of the problem, as well as the existence and uniqueness of a solution, are considered. It is shown that nonlocal conditions allow for a more adequate description of processes in mathematical physics in which the state of a system depends on the distributed characteristics of the environment.

Key words: hyperbolic equation, nonlocal boundary value problem, third-order equation, mathematical physics, well-posedness of the problem, integral conditions.

Boundary and initial conditions. When mathematically describing a physical process, one must first formulate the problem, i.e., formulate conditions sufficient for the unambiguous definition of the process.

Differential equations with ordinary and, especially, partial derivatives, generally have an infinite number of solutions. Therefore, when a physical problem is reduced to a partial differential equation, to unambiguously characterize the process, it is necessary to add certain additional conditions to the equation.

Let's consider the simplest problem of transverse vibrations of a string fixed at its ends. In this problem, $u(x,t)$ the string's deviation from the axis is determined by x. If the string's ends $0 \quad x \quad l$ are fixed, then the "boundary conditions" must be satisfied.

$$u(0,t)=0, \qquad u(l,t)=0. \qquad (1)$$

Since the process of string vibration depends on its initial shape and velocity distribution, the initial “initial conditions” should be set:

$$u(x,t_0)=\varphi(x),$$
$$u_t(x,t_0)=\psi(x).$$

Thus, additional conditions consist of boundary and initial conditions, where $\varphi(x)$ and $\psi(x)-$ are given point functions.

Let us now turn to the consideration of limiting cases of the problem posed. The influence of boundary conditions at a point M_0, sufficiently distant from the boundary at which they are specified becomes apparent after a fairly long period of time.

If we are interested in a phenomenon over a short period of time, when the influence of boundaries is still insignificant, then instead of the complete problem we can consider a limit problem with initial conditions for an unbounded region:

find a solution to the equation

$$u_{tt} = a^2 u_{xx} + f(x,t) \text{ For} - \quad < x < \quad , \quad t > 0,$$

with initial conditions

$$\begin{aligned} u(x,0) &= \varphi(x), \\ u_t(x,0) &= \psi(x) \end{aligned} \quad \text{at} - \quad < x < \quad .$$

This problem is often called the Cauchy problem.

If we are studying a phenomenon near one boundary and the influence of the boundary regime at the second boundary is not significant over the period of time of interest to us, then we come to the formulation of the problem on a semi-bounded line when, in addition to the equation, additional conditions are given:

$$\begin{aligned} u(0,t) &= \mu(t), \\ u(x,0) &= \varphi(x) \\ u_t(x,0) &= \psi(x) \end{aligned} \qquad \begin{aligned} t \quad 0, \\ 0 \quad x < \quad . \end{aligned}$$

The nature of the phenomenon at time instants sufficiently distant from the initial instant $t = 0$, is completely determined by the boundary values, since the influence of the initial conditions, due to the friction inherent in any real system, weakens over time. Problems of this type are especially common in cases where the system is excited by a periodic boundary regime that operates over a long period of time. Such problems "without initial conditions" are formulated as follows:

find a solution to the equation being studied for $0 \quad x \quad l$ at $t > -$ under boundary conditions

$$\begin{aligned} u(0,t) &= \mu_1(t), \\ u(l,t) &= \mu_2(t). \end{aligned}$$

The problem without initial conditions for a semi-bounded line is posed similarly.

If the solution of a mathematical problem continuously depends on additional conditions (on the initial and boundary data and on the right-hand side of the equations – on the initial data of the problem), then the problem is stable.

In the theory of differential equations , initial and boundary conditions are an addition to the main differential equation (ordinary or partial), which specifies its behavior at the initial moment of time or at the boundary of the region under consideration, respectively.

Typically, a differential equation has not a single solution, but a whole family of them. Initial and boundary conditions allow one to be selected, corresponding to a real physical process or phenomenon. In the theory of ordinary differential equations, a theorem on the existence and uniqueness of a solution to a problem with an initial condition (the so-called Cauchy problem) has been proven. For partial differential equations, certain existence and uniqueness theorems have been obtained for certain classes of initial and boundary value problems.

Sometimes, initial conditions in non-stationary problems, such as solving hyperbolic or parabolic equations , are also considered boundary conditions.

Correctness of boundary conditions. For some problems, boundary conditions are divided into principal and natural ones . Principal conditions usually have the form $u(\partial\Omega) = g$, where $\partial\Omega -$ is the boundary of the domain Ω. The natural conditions also contain the derivative of the solution along the normal to the boundary.

Problems in mathematical physics describe real physical processes, and therefore their formulation must satisfy the following natural requirements:

1. The solution must exist in some class of functions;
2. The solution must be unique in some class of functions;
3. The solution must depend continuously on the data (initial and boundary conditions, free term, coefficients, etc.).

Let two differential equations be given: $Lu = F_1, \quad Lu = F_2$ with the same differential operators and the same boundary conditions, then their solutions will continuously depend on the free term if: $\forall \varepsilon > 0 \exists \delta > 0 : \left(\|F_1 - F_2\| < \delta\right) \quad \left(\|u_1 - u_2\| < \varepsilon\right)$, where $u_1, u_2 -$ solutions of the corresponding equations. The incorrect formulation of boundary conditions is well illustrated by Hadamard's example .

Indeed, consider the following Hadamard example: find a regular in the domain $\Omega = \left\{(x,y) : -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad 0 < y < T\right\}$ the solution $u = u(x,y)$ of the Laplace equation is continuous in $\overline{\Omega}$ and satisfies the conditions

$$u(-\frac{\pi}{2}, y) = u(\frac{\pi}{2}, y) = 0, \qquad 0 \quad y \quad T,$$

$$u(x,0) = 0, \qquad u_y(x,0) = \exp(-\sqrt{k})coskx.$$

It is known that the only solution to this local problem has the form

$k = 1,2,...$ (2)

It's easy to see that if , $k \to +$, then the function $exp(-\sqrt{k})\cos kx$ uniformly approaches zero, and not only itself, but also all its derivatives. However, solution (2) for any λ y 0 has the form of a cosine curve with an arbitrarily large amplitude.

The concept of a well-posed problem. The essence of any local or non-local problem P lies in finding its solution u E_u given the initial data. E_u E_φ – some metric spaces with metrics ρ_u and ρ_φ, which are usually determined by the statement of the problem.

Let the concept of a solution to problem P be defined and each element φ E_φ corresponds to a unique solution $u = R(\varphi)$ E_u.

Problem P is called stable on a pair of spaces (E_u, E_φ), if for any $\varepsilon > 0$ it is possible to specify a number such $\delta(\varepsilon) > 0$, that from the inequality $\rho_\varphi(\varphi_1, \varphi_2)$ $\delta(\varepsilon)$ it follows $\rho_u(u_1, u_2)$ ε, where $u_j = R(\varphi_j)$, u_j E_u, u_j E_φ, $j = 1, 2$.

Problem P is called well-posed on a pair of spaces (E_u, E_φ) (or simply well-posed) if:

1) φ E_φ there is a solution for any u E_u;

2) the solution u is determined unambiguously;

3) it is stable on (E_u, E_φ).

Let:

1) the space of continuous $E_u = C[0,T]$ – functions on a segment with a metric 0 t T

$\rho_u(u_1, u_2) = \max_{[0,T]} |u_1(t) - u_2(t)|$ $\|u_1 - u_2\|$;

2) $E_\varphi = R$ – a space with a metric $\rho_\varphi(\varphi_1, \varphi_2) = |\varphi_1 - \varphi_2|$. As a simple example $(C[0,T], R)$ of a problem correctly posed on a pair of spaces, we can consider the Cauchy problem: $u(0) = \varphi > 0$ for the Malthusian equation $u' = \varepsilon u$ with a constant growth rate $\varepsilon = \mu > 0$.

Due to $u(t) = u(\tau)\exp[\mu(t - \tau)]$, where τ – any fixed point from $[t_0, T]$, the solution to this problem has the form

$u(t) = \varphi \exp(\mu\ t)$.

Hence,

$$\|u_1 - u_2\| = |\varphi_1 - \varphi_2| \exp(\mu T).$$

A problem (local or non-local) that is not well-posed on a pair of spaces (E_u, E_φ), is called ill-posed (or simply ill-posed).

It should be noted that the definition of an ill-posed problem applies to this pair, (E_u, E_φ), since in other metrics the same problem may turn out to be correct.

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